

Developing the Concept of a Determinant Using DGS

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Abstract: *The paper looks at some of the conceptual difficulties that students have in the courses Linear Algebra and Analytic Geometry. It identifies students' difficulties in understanding the concept definitions and various concept images of a Determinant. It also suggests an approach using the DGS GeoGebra for overcoming these difficulties, which was undertaken with upper-secondary school students. The last Paragraph reports on students' experiences using the suggested GeoGebra Applet during the experiment.*

1. Introduction

This paper describes computer support in mathematics education and is specified in the field of Linear Algebra and Analytic Geometry. The paper tries to answer the question: How to teach the concept of a Determinant in the upper high schools¹, when students do not have prior knowledge in Vector Spaces, yet it would not possibly cause barriers at the university stage of the courses Linear Algebra and Analytic Geometry. It seeks for the answers on how the students understand this concept and which are their conceptual difficulties (Britton, Henderson 2009). Through provoking questions, it is discovered that students often do not understand the concept definition or create insufficient concept image for this term. Sometimes, not adequate concept image of the term is formed; even they clearly understand the concept definition.

The next Paragraph provides theoretical overview on the concept and contains the provoking questions that motivated this study.

The Paragraph 3 reports on the suggested approach for teaching and learning the introductory level of the term.

The Paragraph 4 shows a teaching experiment and the students' experiences using GeoGebra during the experiment.

For the aims of the study, the Applet [10] was created and it can be found on the web page:

www.math.hu-berlin.de/~filler/donevska/applets .

The reasons for choosing the DGS GeoGebra² are that it is open source software, available on three different platforms and can easily be downloaded. Two advantages of the DGS GeoGebra are that, it is multilingual software (available on more than 50 different languages) and, as its name is a combination of **Geometry** and **Algebra**, it is very suitable to combine topics from both of the mathematics disciplines which are a matter of discussion in the paper.

¹ The upper high schools level in the 11th or 12th grade, namely students at the age of 16 - 18 years.

² <http://www.geogebra.org>

http://en.wikipedia.org/wiki/List_of_interactive_geometry_software#Comparison

2. Theoretical Framework

The concept of a Determinant is important in the upper level of the courses Linear Algebra and Analytic Geometry. Even the introduction to Determinants may sometimes cause problems for different reasons: the concept may be interpreted as a numerical characteristic (value) of a squared matrix or it may be referred to as an algebraic representation of areas and volumes of geometric figures and solids, respectively. Some recourse³ introduce the concept as: “Determinants are mathematical objects that are very useful in the analysis and solutions of systems of linear equations”. What is a Determinant in the student’s mind, actually? What kind of concept images do students build (Harel, 1998) of a Determinant? Is it a number, a squared scheme of numbers, or some kind of a helpful object?

According to Halverscheid [5], there is an algebraic problem (defining a Determinant), which should be solvable by certain algebraic operations and there is the geometric counterpart constructed from a setting in algebra, thus the aim is to understand more about the learning processes when students describe the algebraic operations as a dynamical change of the geometric settings.

Concepts are defined in general terms, but this does not guarantee that students would be able to think about them in general terms (Harel, 1997). This is exactly the case for the concept of a determinant. Students are able to think about it as of a value of two- or three- dimensional matrix, or of an area and volume of two- and three-dimensional geometric shapes (as squares, rectangles and parallelograms; cubes, boxes and parallelepipeds), but the reason for their difficulties with this concept, is in applying its definition to solve problems that involve non-specific geometric figures. Most of the textbooks introduce the term clearly and the students usually have little or no difficulties understanding the meaning of a determinant (that it is actually a number and the procedure of obtaining this number). They can even solve simple problems after the formal definition is introduced. The real difficulties, however, begin when they are asked to solve problems that involve non-regular polygons, such as following:

1. Can a determinant represent the area of a trapezoid?
2. Can a determinant represent the area of a triangle, a pentagon or any convex (or even concave) polygon?
3. Can a determinant represent the volume of any solid?

Most of the students’ answers on these provoking questions are “No”. Their reasons for this answer are the following: the trapezoid is not a parallelogram and the triangle (or the other non-quadrilateral polygons do not have four vertices. This means that they do not have fully developed concept image of a determinant, because they practically establish a relation between the number of the vertices of the geometric figure and the number of the elements in the determinant.

A natural incorporation of DGS would be to help the students in overcoming this kind of dilemmas. For this reason, the role of visualizations of the concept of a determinant is emphasized further on in this paper.

However, at the latter stage⁴ of the courses Linear Algebra and Analytic Geometry, the concept image of a Determinant may change. At this stage, the corresponding matrix is used to represent two linear mappings: one that maps the standard basis vectors to the rows of the matrix, and one that maps them to the columns of the matrix. In either case, the images of the basis vectors form a parallelogram (Task 1 in Paragraph 4.1) that represents the image of the unit square under the mapping. The concept image becomes even more complicated for the determinants of 3 by 3

³ <http://mathworld.wolfram.com/Determinant>

⁴ The later stage means at the university level of the courses Linear Algebra and Analytic Geometry.

matrices, or even n by n matrices, when the visualization is intuitive and very abstract. At this stage, different concept image may be formed if a Determinant is defined using permutations, for example as in [8]. Anyhow, the concept *can* be understood in different ways, it *should* be understood in different ways, and it is *advantageous to change ways of understanding* of a concept while attempting to solve a problem (Harel, 1998).

3. Proposed Approach for Developing the Concept of a Determinant

The approach suggests developing the concept of a Determinant [3] at the beginning (upper-high school) stage, by using the previous students' knowledge (calculating areas of plane figures, such as: square, rectangular, parallelogram, triangle) in 2D Geometry, utilizing the interactive geometry software GeoGebra.

The Applet (Figure 1) provides a support for understanding the concept of a Determinant using the area of a *parallelogram*. The parallelogram $ABCD$ is placed at the Descartes coordinate system, such that the vertex A coincides the origin and the vertices B and D are any other points on the plane. Thus, the coordinates of the vertices of the parallelogram are: $A(0, 0)$, $B(a, b)$, $C(a+c, b+d)$ and $D(c, d)$. The coordinates of the vertices B and D play the crucial role on the Applet. Dragging these points B and D , the student can simultaneously see how the elements of the determinant change. At the same time as the area⁵ of the parallelogram changes, the value of the corresponding determinate, changes as well on the Applet.

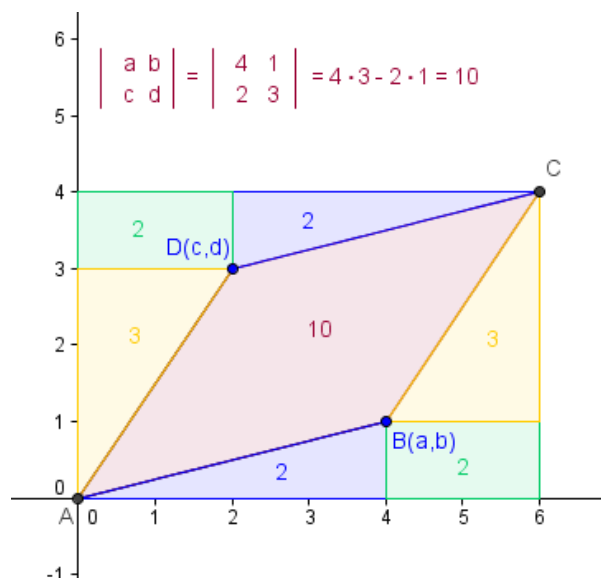


Figure 1. Area of a parallelogram and a triangle using determinants (Applet)

Calculating areas of polygonal regions is not the main focus of the lecture, but teaching and learning the concept of a Determinant. However, the validation that the area of the parallelogram matches the absolute value of the determinant is provided by:

⁵ The area of the geometric figure on the GeoGebra Applet can be displayed by choosing “Value” as a Label of the figure. In this way the unnecessary calculations of areas by hand can be avoided.

$$A_{ABCD} = \pm \left[(a+c) \cdot (b+d) - 2bc - 2\frac{ab}{2} - 2\frac{cd}{2} \right] = \pm(ad - bc) = \pm \begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad (1)$$

This is exactly the definition for the determinant of the two- dimensional matrix:

$$M_{2 \times 2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad (2)$$

where $a, b, c, d \in R$ and the relation between them is:

$$\det M_{2 \times 2} = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad (3)$$

Using the Applet, students can experiment by dragging the vertices B and D along the x - and y -axis respectively, and discover the connections between the areas of the obtained figures (squares and rectangles) and the determinants. They can learn the Identity 2 by 2 matrix and they can observe how changing the order of the vertices B and D effects the sign (+ or -) of the determinant. Further on the concept may be widened to the Determinant of a 3 by 3 matrix by applying the previous students' knowledge in 3D Geometry in calculating volumes of cubes, cuboids, and parallelepipeds and the use of 3D GeoGebra. Thus, the volume of the *parallelepiped* with vertices $A(0,0,0)$, $B(a,b,c)$, $D(d,e,f)$, $A'(g,h,i)$ is:

$$V = \pm \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad (4)$$

Another way to expand the concept to a determinant of a 3 by 3 matrix is shown in Task 3, Paragraph 4.1, where the aim is to introduce the terms for minors and cofactors of a Determinant.

4. Teaching Experiment

The teaching experiment using the Applet was undertaken with a group of 21, 12th grade students in the Käthe Kollwitz Oberschule in Berlin, Germany.

At the beginning of the study, students were asked to define, or if not possible, to describe the term of a determinant. Two of the students in the class wrote the following:

Student 1: "Special $\begin{vmatrix} x & y \\ z & t \end{vmatrix}$ which helps to find solutions".

Student 2: "Number which determinates if a matrix has a solution or not".

Although the students had previous knowledge in Matrices, they did not clearly distinguish the term for a matrix from the term for a determinant. Student 1 used the correct notation for a determinant, but did not refer to it as being a number and how this number is calculated, but rather

to its application, probably meaning to find solutions of systems of linear equations. Student 2 defined it as a number, also without explaining how this number is obtained, but obviously had some misunderstandings about “matrices having solutions”.

After that, the introduction to the concept of a Determinant started with the GeoGebra Applet. On the beginning, the Applet is set up in the position 1, and then students have to change it to the positions 2 and 3 (as shown on the Figure 2); and finally to the position 4 (Figure 1).

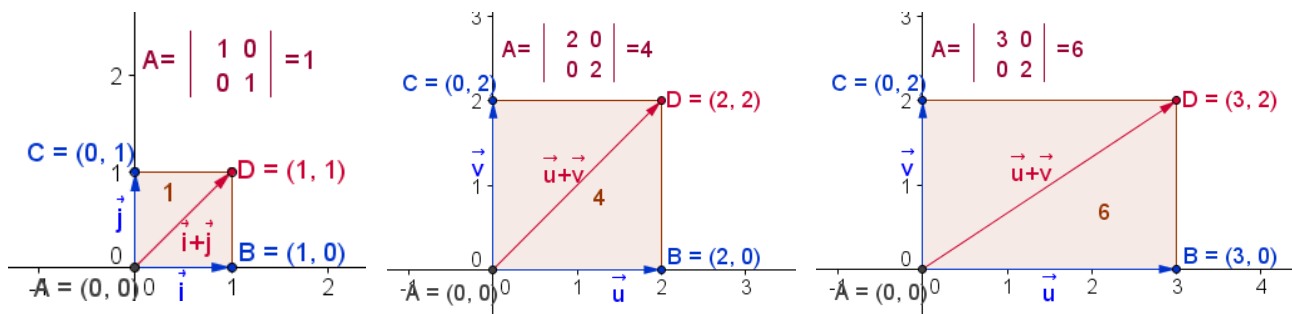


Figure 2. Positions 1, 2 and 3 of the GeoGebra Applet

Students are guided to pay attention on the influence of the position-change of the vertices B and D of the quadrilaterals, on the value of the determinant. They are asked to fill in the three right columns of the Table 1. The accent during the study should be on students work and effort to derive (1) by themselves using the Applet (as the experiment has proved, they are capable of doing so).

<i>Geometric Figure</i>	<i>Vertices</i>	<i>Matrix</i>	<i>Determinant Representing the Area</i>
Square $ABCD$ with side 1	$B(1, 0), D(0, 1)$	$I_{2 \times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$
Square $ABCD$ with side a	$B(a, 0), D(0, a)$	$M_{2 \times 2} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$	$\begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix} = a^2$
Rectangle $ABCD$ with sides a and b	$B(a, 0), D(0, b)$	$M_{2 \times 2} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$	$\pm \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = \pm ab$
Triangle ABD	$B(a, b), D(c, d)$	$M_{2 \times 2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$	$\pm \frac{1}{2} \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \pm \frac{1}{2}(ad - bc)$

Table 1. Connections between Areas of Plane Geometric Figures and 2 by 2 Matrices and Determinants

The aim of filling in the columns of Table 1 is to introduce the students with the notation of the concept. Analogue exercises and Table may be given to the students for the Determinants of 3 by 3 matrices.

4.1 Students' Experiences

After the students were introduced with the proposed approach, they were given some tasks. This paragraph reports on the students' experiences and their results.

Task1. Provide a visualization of the determinant $\begin{vmatrix} 5 & -2 \\ 2 & 3 \end{vmatrix}$ and explain the relation between the given determinant and the area of the geometric figure you have presented.

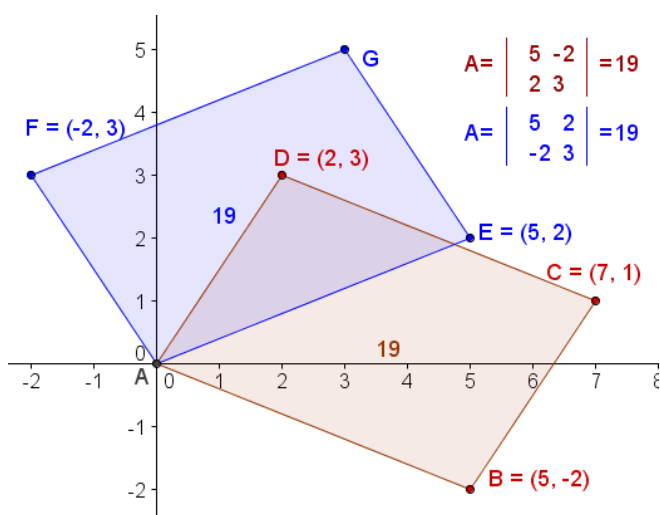


Figure 3. The area of the parallelogram corresponding to the given determinant

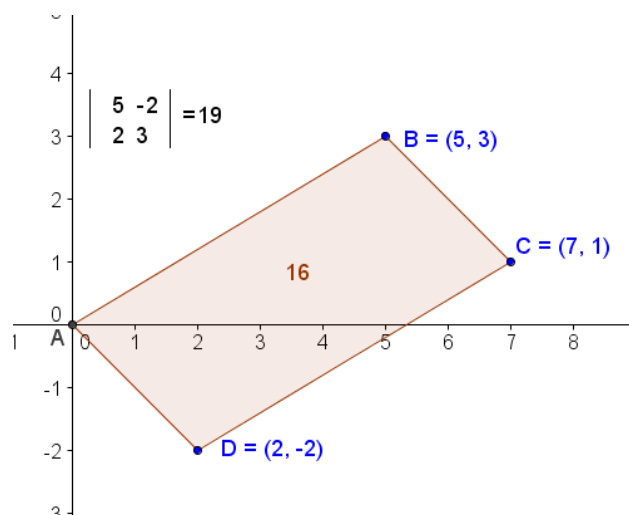


Figure 4. Students' mistake in visualization

The solution of the Task 1 is provided on the Figure 3, which actually provides visualization of the property:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}. \quad (5)$$

The two determinants, $\begin{vmatrix} 5 & -2 \\ 2 & 3 \end{vmatrix}$ and $\begin{vmatrix} 5 & 2 \\ -2 & 3 \end{vmatrix}$, represent the vertices of two different parallelograms

$ABCD$ and $AEGF$ with equal areas of 19 squared units. Most of the students noticed the fact that the given determinant provides areas of these two different parallelograms, thus were able to discover the property. Nevertheless, some students denoted the coordinates of the vertices B and D of the corresponding parallelogram by the elements in the diagonals (Figure 4) instead of the elements in

the rows (columns) (Figure 3). Soon, with the aid of GeoGebra⁶, they realized that the area of such defined parallelogram does not match the value of the given determinant and started searching for the right solution.

Task 2. Write the area of a given trapezoid using determinants.

Even though their first answer was that it is impossible (recall the provoking question 1), students were encouraged to use GeoGebra tools and come up with a solution of the problem. The following three figures show some of their answers and different ways of thinking.

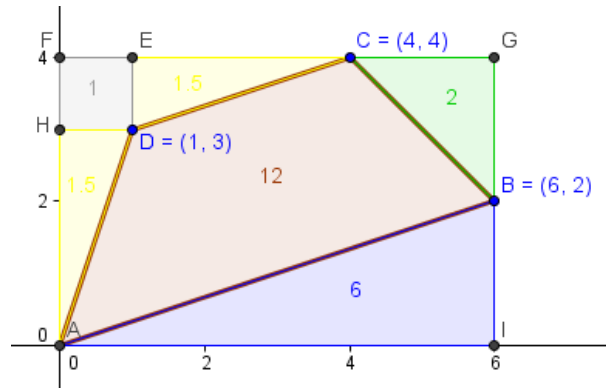


Figure 5. Area of a trapezoid using areas of right-angled triangles and a square

Most of the students, as expected, presented the area as on the Figure 5, but soon they realized that it was difficult for them to present the area of the triangle BGC for example, by determinants, because none of its vertices is at the origin. So, at this stage of their learning process students do not know how to write the area of a triangle if none of its vertices is at the origin, but were able to describe what they would do if they could, for example move the triangle BGC such that the vertex G coincides the origin. This indicates their understanding of the procedure better than carrying out the procedure successfully. Thus, they needed more effort and other GeoGebra tools to search for the precise answer.

Some students realized that it is possible to express the area of the given trapezoid as a sum of the areas of two triangles (Figure 6), whose area is very easy to find using half of the determinants describing the areas of the corresponding parallelograms as:

$$A = \frac{1}{2} \left(\begin{vmatrix} 6 & 2 \\ 4 & 4 \end{vmatrix} + \begin{vmatrix} 4 & 4 \\ 1 & 3 \end{vmatrix} \right) = 12 \text{ cm}^2.$$

⁶ The area of the parallelogram is displayed on the parallelogram in the GeoGebra file and can be directly compared with the value of the determinant.

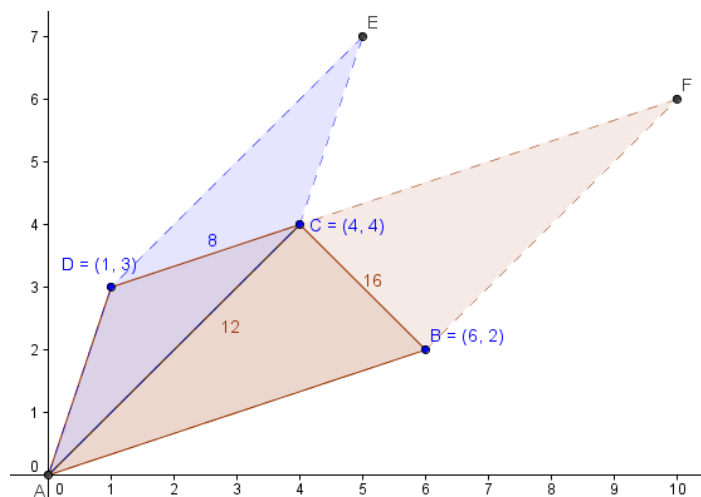


Figure 6. Area of a trapezoid as a sum of the areas of two triangles

In general, for $A(0,0)$, $B(a,b)$, $C(c,d)$ and $D(e,f)$, $a,b,c,d,e,f \in R$, the area of the trapezoid $ABCD$ is:

$$A = \frac{1}{2} \left(\begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} c & d \\ e & f \end{vmatrix} \right).$$

These students were able to connect their visualized solution with the algebraic symbolism, as shown in the last two formulas.

The most elegant solution was obtained using student's previous knowledge in geometric transformations: rotation and translation, as shown on Figure 7. This student applied a composition of one rotation with vertex at the point B for an angle of 180° and one translation for the vector \vec{u} of the given trapezoid. In this way the obtained figure $AD''A''D$ is a parallelogram consisted of two congruent trapezoids (the student was able to prove). The area of the trapezoid $ABCD$, in this case is half of the area of the parallelogram $AD''A''D$ and it can be expressed with only one determinant as,

$$A = \frac{1}{2} \begin{vmatrix} 9 & 3 \\ 1 & 3 \end{vmatrix} = 12 \text{ cm}^2.$$

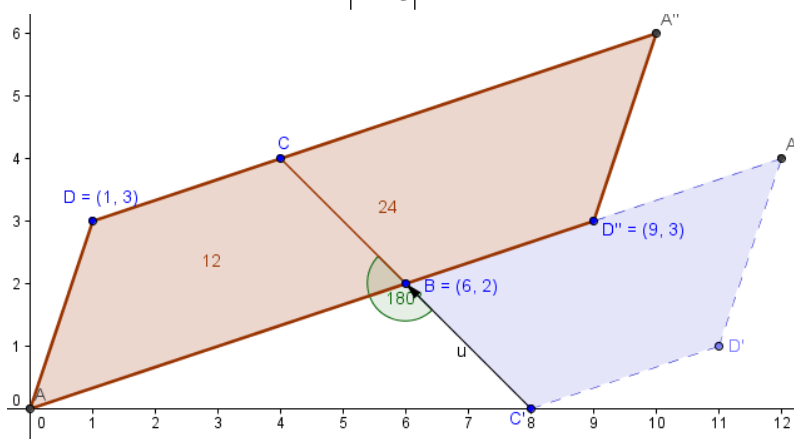


Figure 7. Area of a trapezoid as a half of the area of the parallelogram

After the solution for the area of the particular trapezoid, the student wrote the area of a trapezoid $ABCD$, where $A(0,0)$, $B(a,b)$, $C(c,d)$ and $D(e,f)$, $a,b,c,d,e,f \in R$ as:

$$A = \frac{1}{2} \begin{vmatrix} a+c-e & b+d-f \\ e & f \end{vmatrix},$$

confirming that he had understood the generalization process and the new symbolism. A student with an effective concept image is one who can communicate its corresponding concept definition in her or his words, can think about it in general terms, can connect it to other concepts, and can, as a result, remember its meaning for an extended period of time (Harel, 1998).

Task 3. Find the area of the triangle such that none of its vertices coincide the origin of the Descartes coordinate system.

The vectors were of course not given on the Figure 8, but most of the students came to an idea to divide the given triangle into three other triangles (each one having a vertex at the origin) and find the sum of their areas.

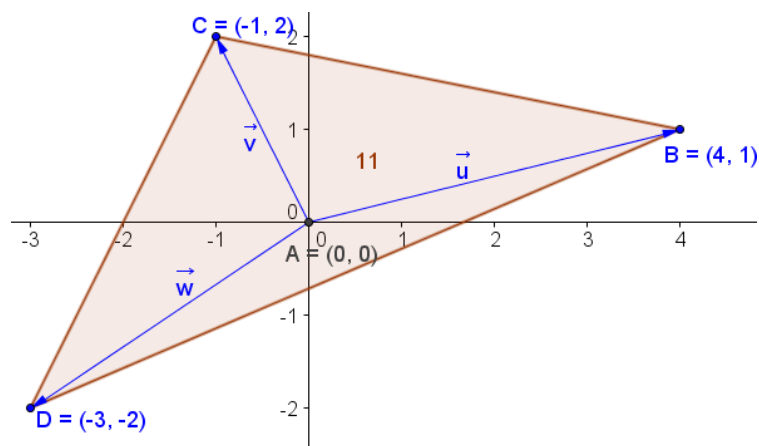


Figure 8. Area of a triangle using determinants

Area of a triangle whose vertices are not points on the coordinate axes, but arbitrary points $B(a,b)$, $C(c,d)$, $D(e,f)$ on the plane can be found by the formula:

$$A_{\triangle BCD} = \frac{1}{2} \left(\begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} c & d \\ e & f \end{vmatrix} + \begin{vmatrix} e & f \\ a & b \end{vmatrix} \right)$$

This particular task was given to the students, because it offers a connection from determinants of 2 by 2 to 3 by 3 matrices, using the following relation:

$$A_{\triangle BCD} = \frac{1}{2} \begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$$

This task shows that there exists a connection between the areas of plane geometric figures and determinants of higher order matrices. Thus, the concept for a determinant of a 3 by 3 matrix should not be restricted to volumes of solids only, but it may be referred to as a number defining the area of plane geometric figure, as well.

Nevertheless some students' misunderstandings occurred during the experiment, (for example, some students thought that if the geometric figure is in the first quadrant of the coordinate system, than the corresponding determinant is always a positive number and if it is in the second quadrant, than the determinant is negative). In fact the sign of the determinant does not depend on the quadrant where the figure is drawn, but on the counterclockwise (+) or clockwise (-) direction of placing the coordinates of the points in the determinant. Students convinced themselves in this fact by investigations with the Applet.

5. Summary

According to Dorier [4], the theory of Linear Algebra is well developed, those who teach it know it personally very well...yet the students do not understand. The teaching experiment in this study proved that students sometimes do not build effective concept images in Linear Algebra and Analytic Geometry courses; rather they place their full reliance on concept definitions, by memorizing them verbatim (Harel, 1997). The paper offers an approach for understanding the concept for Determinants with visualizations in GeoGebra. In the application of the proposed approach the accent is on activating previous students' knowledge in plane and solid geometry in gaining new knowledge in Linear Algebra and Analytic Geometry. The use of the provoking questions in the suggested teaching approach allows two-way communication between the instructor and the students as well as the use of the GeoGebra Applet motivates learning and keeps students engaged (Chang, 2011). One more advantage of the applied GeoGebra Applet is that students can see the algebraic expressions and the visualized representation of the concept *simultaneously*, thus can form a correct concept image at the beginning stage of Linear Algebra and Analytic Geometry courses.

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